लक्ष्यं विद्यामानम्

WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 6th Semester Examination, 2023

MTMACOR14T-MATHEMATICS (CC14)

RING THEORY AND LINEAR ALGEBRA II

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: 2×5 =10

(a) In the ring $\mathbb{Z}_8[x]$, show that $[1]+[2]x$ is a unit.

(b) Let R be an integral domain and p be a prime element in R. Then show that p is irreducible.

(c) Prove that [2] is not an irreducible element in $\mathbb{Z}_6$.

(d) Find all associates of $1+i$ in $Z[i]$.

(e) Let T be a linear operator on $V=\mathbb{R}^2$ defined by $T(a,b)=(-2a+3b,-10a+9b)$. Find the eigen values of T.

(f) Let S be a subset of a vector space V over a field F. Show that S^0 is a subspace of V^*, where S^0 denotes the annihilator of S and V^* denotes the dual space of V.

(g) Let $\langle\,,\rangle$ be the standard inner product on $\mathbb{R}^2$. Let $\alpha=(1,2)$ and $\beta=(-1,1)$. If γ is a vector such that $\langle\alpha,\gamma\rangle=-1$ and $\langle\beta,\gamma\rangle=3$, find γ.

(h) Let V be a finite dimensional vector space. What is the minimal polynomial of the identity operator on V?

2. (a) If D is an integral domain, show that $D[x]$ is an integral domain. Also show that if D is a field, then $D[x]$ can never be a field. 3+1

(b) Let F be a field and $\alpha:F[x]\to F[x]$ be an automorphism such that $\alpha(a)=a$ for all $a\in F$. Show that $\alpha(x)=ax+b$ for some $a,b\in F$. 4

3. (a) Prove that every irreducible element in a UFD is a prime element. 4

(b) Test for the irreducibility of the following polynomials: 2+2

(i) $x^3-[9]$ over $\mathbb{Z}_{11}$

(ii) $x^4+x^3+x^2+x+1$ over $\mathbb{Z}$.

4. (a) Prove that every Euclidean domain is a PID. 4

(b) Find a gcd of the elements $3+i, 5+i$ in the Euclidean domain $Z[i]$ with a Euclidean valuation v defined by $v(m+ni)=m^2+n^2$ for $m+ni \in Z[i]$. If d is the gcd, express d as $d=(3+i)u+(5+i)v$ for some u, v in $Z[i]$. 4

5. (a) Find the dual basis for the ordered basis 4

$$\mathcal{B}=\{(1,-2,3),(1,-1,1),(2,-4,7)\} \text{ of } V_3(\mathbb{R}).$$

(b) Let V be a finite dimensional vector space over a field $\mathbb{F}$. Define $\psi: V \to V^{**}$ by $\psi(x)=\hat{x}, \forall x \in V$, where $\hat{x}: V^* \to \mathbb{F}$ is defined by $\hat{x}(f)=f(x), \forall f \in V^*$. Show that ψ is an isomorphism from V to V^{**}. 4

6. (a) Apply Gram-Schmidt process to the vectors $\beta_1=(1,0,1)$, $\beta_2=(1,0,-1)$, $\beta_3=(0,3,4)$, to obtain an orthonormal basis of $\mathbb{R}^3$ with the standard inner product. 4

(b) Find a matrix P such that $P^{-1}AP$ is in Jordan Canonical form, where 4

$$A=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

7. (a) Let W be a subset of a vector space V over a field K. Define the *annihilator* of W. If U and W are subspaces of a vector space V over a field K, then show that $(U+W)^0=U^0 \cap W^0$, where U^0, W^0 are annihilators of U, W respectively. 1+4

(b) If W is a subspace of R^4, generated by (1, 2, 3, 4) and (1, 1, 1, 1), then find a basis of the annihilator of W. 3

8. (a) Let T be a linear operator on a complex inner product space V. Prove that T is normal if and only if $\|T^*(u)\|=\|T(u)\|, \ \forall u \in V$. 4

(b) Prove that the matrix $A=\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable over the field C. 4

9. Let $V=P_3(\mathbb{R})$ and T be a linear operator on V defined by $T(p(x))=xp'(x)+p''(x)-p(2)$. $\mathcal{B}=\{1, x, x^2, x^3\}$ is the standard ordered basis for V. Find the matrix representation of T relative to the basis $\mathcal{B}$ for V. Find the characteristic polynomial of $[T]_{\mathcal{B}}$. Show that T is diagonalizable. Also find the minimal polynomial of T. 1+2+4+1

(Here $p'(x)$ and $p''(x)$ denote the first and second order derivative of $p(x)$)

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